5. $f(x) = \begin{cases} a + bx^2, & 0 \leq x \leq 1 \\ 0, & w.p.p. \end{cases}$

Wiemy, że $\int_0^1 f(x)\,dx = 1$ i $\int_0^1 x \cdot f(x)\,dx = E(X) = 0,6$.

$$\int_0^1 f(x)\,dx = \int_0^1 a + bx^2\,dx = \left[ax + \frac{bx^3}{3}\right]_0^1 = a + \frac{b}{3}$$

$$\int_0^1 x \cdot f(x)\,dx = \int_0^1 ax + bx^3\,dx = \left[\frac{ax^2}{2} + \frac{bx^4}{4}\right]_0^1 = \frac{a}{2} + \frac{b}{4}$$

A więc:

$$\begin{cases} a + \frac{b}{3} = 1 \\ \frac{a}{2} + \frac{b}{4} = \frac{3}{5} \end{cases} \iff \begin{cases} 3a + b = 3 \\ 10a + 5b = 12 \end{cases} \iff \begin{cases} b = 3 - 3a \\ 10a + 15 - 15a = 12 \end{cases} \iff \begin{cases} b = 3 - 3a \\ a = \frac{3}{5} \end{cases} \iff \begin{cases} a = \frac{3}{5} \\ b = \frac{6}{5} \end{cases}$$

6. $\Gamma(p) = \int_0^\infty x^{p-1} e^{-x}\,dx$. Wykaż, że $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. Wsk. Podst. $y = \sqrt{2x}$

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty x^{-\frac{1}{2}} e^{-x}\,dx = \int_0^\infty \frac{1}{\sqrt{x}} e^{-x}\,dx = \left\{\begin{array}{l} y = \sqrt{2x} \implies x = \frac{1}{2}y^2 \\ dy = \frac{1}{\sqrt{2x}}\,dx \end{array}\right\} = \int_0^\infty \frac{1}{\sqrt{\frac{1}{2}y^2}} e^{-\frac{1}{2}y^2} \sqrt{2 \cdot \tfrac{1}{2}y^2}\,dy =$$

$$= \sqrt{2} \int_0^\infty e^{-\frac{1}{2}y^2}\,dy = (*)$$

Wiemy, że $\int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2}\,dy = \sqrt{2\pi}$ (lista nr 2). Zauważ, że $e^{-\frac{1}{2}y^2}$ jest funkcją parzystą, bo $f(-y) = e^{-\frac{1}{2}(-y)^2} = e^{-\frac{1}{2}y^2} = f(y)$. A więc $\int_0^\infty e^{-\frac{1}{2}y^2}\,dy = \frac{1}{2}\int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2}\,dy = \frac{1}{2}\sqrt{2\pi}$.

Mamy więc:

$$(*) = \sqrt{2} \cdot \tfrac{1}{2}\sqrt{2\pi} = \sqrt{\pi} \quad \blacksquare$$

9. X i Y są niezależne, a więc $f(x,y) = f_X(x) \cdot f_Y(y) = -6x(x-1)-6y(y-1) = 36x(x-1)y(y-1)$

Z zad. 8 wiemy, że:
- $x < y \implies x \in \left(0, \frac{1}{2}\right), y \in \left(\frac{1}{2}, x + \frac{1}{2}\right)$
- $x > y \implies x \in \left(\frac{1}{2}, 1\right), y \in \left(x - \frac{1}{2}, \frac{1}{2}\right)$

Czyli trzeba policzyć:

$$\int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^{x+\frac{1}{2}} f(x,y)\,dy\,dx + \int_{\frac{1}{2}}^{1} \int_{x-\frac{1}{2}}^{\frac{1}{2}} f(x,y)\,dy\,dx \overset{(1),(2)}{=} \frac{61}{320} + \frac{61}{320} = \frac{61}{160}$$

1) $$\int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^{x+\frac{1}{2}} 36x(x-1)y(y-1)\,dy\,dx = 36 \int_0^{\frac{1}{2}} x(x-1) \left(\int_{\frac{1}{2}}^{x+\frac{1}{2}} y^2 - y\,dy\right) dx = 36 \int_0^{\frac{1}{2}} x(x-1) \left[\frac{y^3}{3} - \frac{y^2}{2}\right]_{\frac{1}{2}}^{x+\frac{1}{2}} dx =$$

$$= 36\int_0^{\frac12} x(x-1)\left[\frac{y^2}{6}(2y-3)\right]_{\frac12}^{x+\frac12} dx = 36\int_0^{\frac12}(x^2-x)\left(\frac16\left(x+\frac12\right)^2(2x+1-3) - \frac16\cdot\frac14(1-3)\right) dx =$$

$$= 36\int_0^{\frac12}(x^2-x)\left(\frac16\left(x^2+x+\frac14\right)(2x-2)+\frac1{12}\right) dx = 36\int_0^{\frac12}(x^2-x)\left(\frac4{12}\left(x^3-\frac34x-\frac14\right)+\frac1{12}\right) dx =$$

$$= 36\int_0^{\frac12}(x^2-x)\,\frac1{12}(4x^3-3x)\,dx = 3\int_0^{\frac12}(4x^5-3x^3-4x^4+3x^2)\,dx =$$

$$= 3\left[\frac46x^6-\frac34x^4-\frac45x^5+x^3\right]_0^{\frac12} = 3\left(\frac23\cdot\left(\frac12\right)^6-\frac34\cdot\left(\frac12\right)^4-\frac45\left(\frac12\right)^5+\frac18\right) =$$

$$= 3\left(\frac1{96}-\frac3{64}-\frac1{30}+\frac18\right) = 3\cdot\frac{53}{960} = \frac{53}{320}$$ // WOLF. $\to \frac{61}{320}$

2) $$\int_{\frac12}^{1}\int_{x-\frac12}^{\frac12} 36x(x-1)y(y-1)\,dy\,dx = 36\int_{\frac12}^{1}x(x-1)\left(\int_{x-\frac12}^{\frac12}y^2-y\,dy\right)dx = 36\int_{\frac12}^{1}x(x-1)\left[\frac{y^3}{3}-\frac{y^2}{2}\right]_{x-\frac12}^{\frac12}dx =$$

$$= 36\int_{\frac12}^{1}(x^2-x)\frac16\left[y^2(2y-3)\right]_{x-\frac12}^{\frac12}dx = 6\int_{\frac12}^{1}(x^2-x)\left(\frac14(1-3)-\left(x-\frac12\right)^2(2x-1-3)\right)dx =$$

$$= 6\int_{\frac12}^{1}(x^2-x)\left(-\frac12-\left(x^2-x+\frac14\right)(2x-4)\right)dx = 6\int_{\frac12}^{1}(x^2-x)\left(-2x^3+6x^2-\frac92x+\frac12\right)dx =$$

$$= 6\int_{\frac12}^{1}-2x^5+6x^4-\frac92x^3+\frac12x^2+2x^4-6x^3+\frac92x^2-\frac12x\,dx = 6\int_{\frac12}^{1}-2x^5+8x^4-\frac{21}{2}x^3+5x^2-\frac12x\,dx =$$

$$= 6\left[-\frac26x^6+\frac85x^5-\frac{21}{8}x^4+\frac53x^3-\frac{x^2}{4}\right]_{\frac12}^{1} = 6\left(-\frac13+\frac85-\frac{21}{8}+\frac53-\frac14+\frac1{192}-\frac1{20}+\frac{21}{128}-\frac5{24}+\frac1{16}\right)$$

$$= 6\cdot\left(\frac{61}{1920}\right) = \frac{61}{320}$$

10.

(X_1, X_2) – zmienna o gęstości $h(x_1,x_2) = \frac1\pi$, $0 < x_1^2+x_2^2 < 1$.

$\Rightarrow$ $x_1 = 0 \wedge (-1 < x_2 < 0 \vee 0 < x_2 < 1)$

$(-1 < x_1 < 0 \vee 0 < x_1 < 1) \wedge \left(-\sqrt{1-x_1^2} < x_2 < \sqrt{1-x_1^2}\right)$

Czyli mamy ~~całki~~ całki:

$$\int_{-1}^{0}\int_{-\sqrt{1-x_1^2}}^{\sqrt{1-x_1^2}}\frac1\pi\,dx_2\,dx_1 + \int_{0}^{1}\int_{-\sqrt{1-x_1^2}}^{\sqrt{1-x_1^2}}\frac1\pi\,dx_2\,dx_1 = \int_{-1}^{1}\int_{-\sqrt{1-x_1^2}}^{\sqrt{1-x_1^2}}\frac1\pi\,dx_2\,dx_1$$, bo jest ciągła

w 0 $\left(\lim_{x_1\to 0^-}\sqrt{1-x_1^2} = \lim_{x_1\to 0^+}\sqrt{1-x_1^2} = 1\right)$.

Można sprawdzić, czy całkuje się do jedynki:

$$\int_{-1}^{1}\int_{-\sqrt{1-x_1^2}}^{\sqrt{1-x_1^2}}\frac1\pi\,dx_2\,dx_1 = \frac2\pi\int_{-1}^{1}\sqrt{1-x_1^2}\,dx_1 = \left\{\begin{array}{l} x_1 = \sin y \Rightarrow y = \arcsin x_1 \\ dx_1 = \cos y\,dy \end{array}\right\} = \frac2\pi\int_{-\frac\pi2}^{\frac\pi2}\cos^2 y\,dy =$$

$$= \frac2\pi\int_{-\frac\pi2}^{\frac\pi2}\frac{1+\cos(2y)}{2}\,dy = \frac1\pi\left(\int_{-\frac\pi2}^{\frac\pi2}1\,dy+\int_{-\frac\pi2}^{\frac\pi2}\cos(2y)\,dy\right) = \frac1\pi\left(\pi+\left[\frac{\sin(2y)}{2}\right]_{-\frac\pi2}^{\frac\pi2}\right) = \frac1\pi(\pi+0-0) = 1$$ ok.