RPIS - LISTA 6

1.
$$E\left(\frac{1}{X+1}\right) = \sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k}p^k(1-p)^{n-k} \overset{(q=1-p)}{=\!=\!=} \sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}p^k q^{n-k} \overset{(1)}{=} \sum_{k=0}^{n}\frac{1}{n+1}\binom{n+1}{k+1}p^k q^{n-k}$$

$$= \sum_{\substack{0\le k\le n \\ 1\le \underbrace{k+1}_{=l}\le n+1}} \frac{1}{n+1}\binom{n+1}{l}p^{l-1}q^{n-(l-1)} = \frac{1}{(n+1)p}\sum_{l=1}^{n+1}\binom{n+1}{l}p^l q^{(n+1)-l} \overset{(2)}{=}$$

$$= \frac{(p+q)^{n+1}-q^{n+1}}{(n+1)p} = \frac{(p+1-p)^{n+1}-(1-p)^{n+1}}{(n+1)p} = \frac{1-(1-p)^{n+1}}{(n+1)p}$$

(1):
$$\frac{1}{k+1}\binom{n}{k} = \frac{1}{k+1}\cdot\frac{n+1}{n+1}\cdot\frac{n!}{k!(n-k)!} = \frac{1}{n+1}\cdot\frac{(n+1)!}{(k+1)!(n+1-k-1)!} = \frac{1}{n+1}\cdot\binom{n+1}{k+1}$$

(2):
$$\sum_{l=1}^{n+1}\binom{n+1}{l}p^l q^{(n+1)-l} = \sum_{l=0}^{n+1}\binom{n+1}{l}p^l q^{(n+1)-l} - \binom{n+1}{0}p^0 q^{(n+1)-0} = (p+q)^{n+1} - q^{n+1}$$

2.
Rozkład Poissona:
$$f(x) = e^{-\lambda}\cdot\frac{\lambda^k}{k!}, \quad k=0,1,2,\dots$$

$$E(X^n) = \sum_{k=0}^{\infty} k^n\cdot e^{-\lambda}\frac{\lambda^k}{k!} = \sum_{k=1}^{\infty} k^{n-1}\cdot e^{-\lambda}\frac{\lambda^{k-1}}{(k-1)!}\cdot\lambda = \lambda\sum_{\substack{1\le k\le\infty \\ 0\le \underbrace{k-1}_{=l}\le\infty}} (l+1)^{n-1}e^{-\lambda}\frac{\lambda^l}{l!} =$$

$$= \lambda\cdot E\left((X+1)^{n-1}\right)$$

$$\left|\left|\ \sum_{k=0}^{\infty}\frac{a^k}{k!} = e^a\right.\right.$$

Z tego:

$$E(X) = \lambda \quad (\text{lista } 2):\ E(X) = \sum_{k=1}^{\infty} k\cdot e^{-\lambda}\frac{\lambda^k}{k!} = \lambda e^{-\lambda}\sum_{k=1}^{\infty}\frac{\lambda^{k-1}}{(k-1)!} = \lambda e^{-\lambda}\overbrace{\sum_{l=0}^{\infty}\frac{\lambda^l}{l!}}^{=e^\lambda} = \lambda e^{-\lambda}\cdot e^{\lambda} = \lambda$$

$$E(X^2) = \lambda\cdot E(X+1) = \lambda\cdot(E(X)+1) = \lambda(\lambda+1)$$

$$E(X^3) = \lambda\cdot E((X+1)^2) = \lambda\cdot E(X^2+2X+1) = \lambda(E(X^2)+2E(X)+1) =$$

$$= \lambda(\lambda^2+\lambda+2\lambda+1) = \lambda(\lambda^2+3\lambda+1)$$

3.
$$E(X!) = \sum_{k=0}^{\infty} k!\cdot e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}\sum_{k=0}^{\infty}\lambda^k \underset{\substack{\uparrow \\ \text{gdy } 0<\lambda<1}}{=} e^{-\lambda}\cdot\frac{1}{1-\lambda} = \frac{e^{-\lambda}}{1-\lambda}$$

4.

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}}\,dy\,dx = \left\{\begin{matrix} x = r\cos\theta & y = r\sin\theta \\ dy\,dx = |J|\cdot d\theta\,dr \end{matrix}\right\} = (*)$$

Podstawienie zamienia nam pierwotnie obszar analityczny na parametryczny, gdzie $r \in (0,\infty)$, a $\theta \in (0, 2\pi)$. Liczymy jakobian:

$$|J| = \begin{vmatrix} \frac{dx}{dr} & \frac{dx}{d\theta} \\ \frac{dy}{dr} & \frac{dy}{d\theta} \end{vmatrix} = \begin{vmatrix} \underset{[po\ r]}{(r\cos\theta)'} & \underset{[po\ \theta]}{(r\cos\theta)'} \\ \underset{[po\ r]}{(r\sin\theta)'} & \underset{[po\ \theta]}{(r\sin\theta)'} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r$$

$$(*) = \int_0^{\infty}\int_0^{2\pi} e^{-\frac{r^2\cos^2\theta + r^2\sin^2\theta}{2}}\, r\,d\theta\,dr = \int_0^{\infty} r e^{-\frac{r^2}{2}}\int_0^{2\pi} 1\,d\theta\,dr = 2\pi\int_0^{\infty} r e^{-\frac{r^2}{2}}\,dr =$$

$$= 2\pi\left[-e^{-\frac{r^2}{2}}\right]_0^{\infty} = 2\pi$$

8.

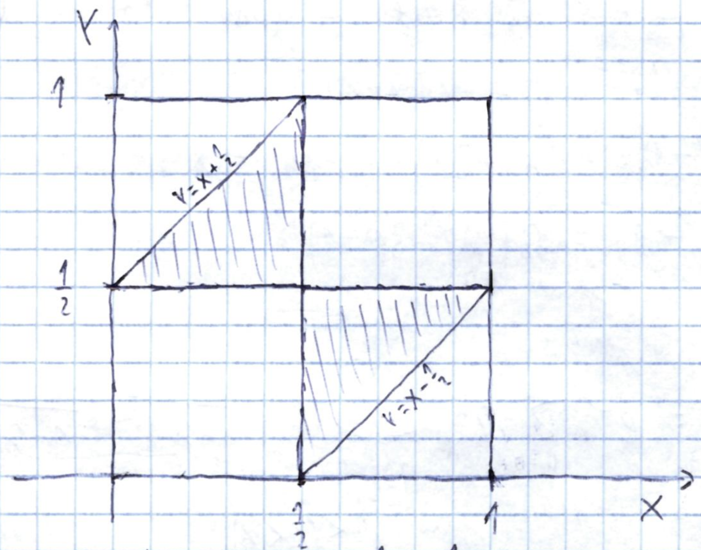
$$f_X(x) = f_Y(x) = \frac{1}{1-0} = 1 \quad , \quad 0 < x, y < 1$$

Mamy dwa przypadki:

- $X < Y$: $\underbrace{0 \;\; X}_{X}\;\underbrace{X \;\; Y}_{Y-X}\;\underbrace{Y \;\; 1}_{1-Y}$
- $Y < X$: $\underbrace{0 \;\; Y}_{Y}\;\underbrace{Y \;\; X}_{X-Y}\;\underbrace{X \;\; 1}_{1-X}$

$$\begin{cases} X < Y - X + 1 - Y \\ Y - X < X + 1 - Y \\ 1 - Y < X + Y - X \end{cases} \Leftrightarrow \begin{cases} X < \frac{1}{2} \\ Y < X + \frac{1}{2} \\ \frac{1}{2} < Y \end{cases} \qquad \begin{cases} Y < X - Y + 1 - X \\ X - Y < Y + 1 - X \\ 1 - X < Y + X - Y \end{cases} \Leftrightarrow \begin{cases} Y < \frac{1}{2} \\ X - \frac{1}{2} < Y \\ \frac{1}{2} < X \end{cases}$$

A więc:



A więc prawdopodobieństwo $p = 2\cdot\frac{1}{8} = \frac{1}{4}$.