LISTA 2

Zadanie 1.

$$\int_x \int_y f(x,y)\,dy\,dx = \int_0^2 \int_0^1 Cxy + x + y\,dy\,dx = \int_0^2 \left[\frac{1}{2}Cxy^2 + xy + \frac{1}{2}y^2\right]_0^1 dx = \int_0^2 \frac{1}{2}Cx + x + \frac{1}{2}\,dx =$$

$$\left[\frac{1}{4}Cx^2 + \frac{1}{2}x^2 + \frac{1}{4}x^2\right]_0^2 = C + 2 + 1 = C + 3$$

$C + 3 = 1 \quad \rightarrow \quad C = -2$

$f(x,y) = -2\mathrm{xy} + x + y$ jest ≤ 0, ponieważ $f(2,1) = -2 \cdot 2 \cdot 1 + 2 + 1 = -1$

Zadanie 2.

a)

$$\int_x \int_y f(x,y)\,dy\,dx = \int_0^2 \int_0^1 Cxy + x\,dy\,dx = \int_0^2 \left[\frac{1}{2}C\,x\,y^2 + xy\right]_0^1 dx = \int_0^2 \frac{1}{2}Cx + x\,dx = \left[\frac{1}{4}C\,x^2 + \frac{1}{2}x^2\right]_0^2 = C + 2$$

$C + 2 = 1 \quad \rightarrow \quad C = -1$

$f(x,y) = -xy + x = x(-y+1)$ jest ≥ 0, ponieważ x $\geq$ 0 oraz y $\leq$ 1

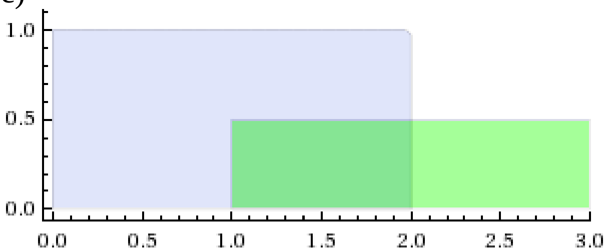
b)

$$F_x(x) = \int_y f(x,y)\,dy = \int_0^1 -xy + x\,dy = \left[-\frac{1}{2}xy^2 + xy\right]_0^1 = -\frac{1}{2}x + x = \frac{x}{2}$$

$$F_y(y) = \int_x f(x,y)\,dx = \int_0^2 -xy + x\,dx = \left[-\frac{1}{2}x^2 y + \frac{x^2}{2}\right]_0^2 = -2\mathrm{y} + 2$$

$$F_x(x)\,F_y(y) = \frac{x}{2} \cdot (-2\mathrm{y} + 2) = \frac{-2\mathrm{yx} + 2\mathrm{x}}{2} = -xy + x = f(x,y)$$

c)



$$P(1 \leq X \leq 3,\ \ 0 \leq Y \leq 0.5) = P(1 \leq X \leq 2,\ \ 0 \leq Y \leq 0{,}5) =$$

$$\int_1^2 \int_0^{0,5} -xy + x\,dy\,dx = \int_1^2 \left[-\frac{1}{2}xy^2 + xy\right]_0^{0,5} dx = \int_1^2 -\frac{1}{8}x + \frac{4}{8}x\,dx = \int_1^2 \frac{3}{8}x\,dx = \left[\frac{3}{16}x^2\right]_1^2 = \frac{3}{16}(4-1) = \frac{9}{16}$$

Zadanie 3.

$$(1)\quad \int y e^{-y}\,dy = -e^{-y} = \begin{vmatrix} f=y & g'=e^{-y} \\ f'=1 & g=-e^{-y} \end{vmatrix} = -ye^{-y} + \int e^{-y} = -y e^{-y} - e^{-y} = -e^{-y}(y+1)$$

$$(2)\quad \int y^2 e^{-y}\,dy = -e^{-y} = \begin{vmatrix} f=y^2 & g'=e^{-y} \\ f'=2\mathrm{y} & g=-e^{-y} \end{vmatrix} = -y^2 e^{-y} + 2\int ye^{-y} = -y^2 e^{-y} - 2e^{-y}(y+1)$$

$$(3)\quad \int y^3 e^{-y}\,dy = -e^{-y} = \begin{vmatrix} f=y^3 & g'=e^{-y} \\ f'=3\mathrm{y}^2 & g=-e^{-y} \end{vmatrix} = -y^3 e^{-y} + 3\int y^2 e^{-y} = -y^3 e^{-y} + 3\left(-y^2 e^{-y} - 2e^{-y}(y+1)\right) =$$
$$-e^{-y}(y^3+3y^2+6y+6)$$

a)

$$\int_x \int_y f(x,y)\,dy\,dx = \int_0^\infty \int_0^\infty C(x+y)e^{-(x+y)}\,dy\,dx = \int_0^\infty \int_0^\infty C x e^{-(x+y)} + C y e^{-(x+y)}\,dy\,dx =$$

$$\int_0^\infty \left(\int_0^\infty C x e^{-x} e^{-y}\,dy + \int_0^\infty C y e^{-x} e^{-y}\,dy \right) dx = \int_0^\infty C e^{-x} \left(x \int_0^\infty e^{-y}\,dy + \int_0^\infty y e^{-y}\,dy \right) dx =$$

$$\int_0^\infty C e^{-x} \left(x \cdot \lim_{\alpha\to\infty} \left(\int_0^\alpha e^{-y}\,dy \right) + \lim_{\alpha\to\infty} \left(\int_0^\alpha y e^{-y}\,dy \right) \right) dx \overset{(1)}{=} \int_0^\infty C e^{-x} \left(x \cdot \lim_{\alpha\to\infty} \left[-e^{-y}\right]_0^\alpha + \lim_{\alpha\to\infty} \left[-e^{-y}(y+1)\right]_0^\alpha \right) dx =$$

$$\int_0^\infty C e^{-x} \left(x \cdot \lim_{\alpha\to\infty} \left(-e^{-\alpha} - (-e^0)\right) + \lim_{\alpha\to\infty} \left(-e^{-\alpha}(\alpha+1) - (-e^0(0+1))\right) \right) dx =$$

$$\int_0^\infty C e^{-x}(x+1)\,dx = C\int_0^\infty e^{-x}(x+1)\,dx = C\int_0^\infty x e^{-x}\,dx + C\int_0^\infty e^{-x}\,dx \overset{\text{to już było liczone}}{=} C\cdot 1 + C\cdot 1 = 2\mathrm{C}$$

$$2\mathrm{C} = 1 \;\rightarrow\; C = \frac{1}{2}$$

$f(x,y) = \frac{1}{2}(x+y)e^{-(x+y)}$ jest ≥ 0, ponieważ x, y > 0

b)

$$f_x(x) = \int_0^\infty \frac{1}{2}(x+y)e^{-x}e^{-y}\,dx \overset{\text{to już było liczone}}{=} \frac{1}{2}e^{-x}(x+1)$$

$$f_y(y) = \int_0^\infty \frac{1}{2}(x+y)e^{-x}e^{-y}\,dx \overset{\text{to już było liczone}}{=} \frac{1}{2}e^{-y}(y+1)$$

$$f_x(x) f_y(y) = \frac{1}{2}e^{-x}(x+1)\frac{1}{2}e^{-x}(y+1) = \frac{1}{4}e^{-(x+y)}(x+1)(y+1) \neq f(x,y)$$

$$m_{pq} = \int_{-\infty}^\infty \int_{-\infty}^\infty x^p y^r f(x,y)\,dy\,dx$$

$$m_{10} = \int_0^\infty \int_0^\infty \frac{1}{2}x(x+y)e^{-(x+y)}\,dy\,dx = \frac{1}{2}\int_0^\infty x \int_0^\infty (x+y)e^{-(x+y)}\,dy\,dx \overset{\text{to już było liczone}}{=} \frac{1}{2}\int_0^\infty x e^{-x}(x+1)\,dx =$$
$$\frac{1}{2}\int_0^\infty x^2 e^{-x}\,dx + \frac{1}{2}\int_0^\infty x e^{-x}\,dx \overset{(1,2)}{=} \frac{1}{2}\left[-x^2 e^{-x} - 2e^{-x}(x+1)\right]_0^\infty + \frac{1}{2}\left[-e^{-x}(x+1)\right]_0^\infty = \frac{1}{2}(2) + \frac{1}{2} = \frac{3}{2}$$

$$m_{01} = \int_0^\infty \int_0^\infty \frac{1}{2}y(x+y)e^{-(x+y)}\,dy\,dx = \frac{1}{2}\int_0^\infty xe^{-x}\left(\int_0^\infty y e^{-y} + y^2 e^{y}\,dy\right) dx =$$
$$\frac{1}{2}\int_0^\infty xe^{-x}\left(\int_0^\infty y e^{-y}\,dy + \int_0^\infty y^2 e^{y}\,dy\right) dx \overset{\text{to już było liczone}}{=} \frac{1}{2}\int_0^\infty xe^{-x}(1+2)\,dx \overset{\text{to już było liczone}}{=} \frac{3}{2}\int_0^\infty xe^{-x}\,dx = \frac{3}{2} = m_{10}$$

$$m_{11}=\int_0^\infty\int_0^\infty \frac{1}{2}xy(x+y)e^{-(x+y)}dy\,dx=\int_0^\infty\int_0^\infty \frac{1}{2}xy(x+y)e^{-x}e^{-y}dy\,dx=\int_0^\infty \frac{1}{2}xe^{-x}\int_0^\infty y(x+y)e^{-y}dy\,dx=$$

$$\frac{1}{2}\int_0^\infty x^2e^{-x}\int_0^\infty ye^{-y}dy\,dx+\frac{1}{2}\int_0^\infty xe^{-x}\int_0^\infty y^2e^{-y}dy\,dx \overset{\text{to już było liczone}}{=} \frac{1}{2}\int_0^\infty x^2e^{-x}dx+\frac{1}{2}\int_0^\infty 2xe^{-x}dx=$$

$$\frac{1}{2}\int_0^\infty x^2e^{-x}dx+\int_0^\infty xe^{-x}dx \overset{\text{to już było liczone}}{=} \frac{1}{2}\cdot 2+1=2$$

c)

$$m_{20}=\int_0^\infty\int_0^\infty \frac{1}{2}x^2(x+y)e^{-(x+y)}dy\,dx=\frac{1}{2}\int_0^\infty x^2\int_0^\infty (x+y)e^{-(x+y)}dy\,dx \overset{\text{to już było liczone}}{=} \frac{1}{2}\int_0^\infty x^2e^{-x}(x+1)dx=$$

$$\frac{1}{2}\int_0^\infty x^3e^{-x}dx+\frac{1}{2}\int_0^\infty x^2e^{-x}dx \overset{(3),\text{ to już było liczone}}{=} =\frac{1}{2}\left[-e^{-y}(y^3+3y^2+6y+6)\right]_0^\infty+\frac{1}{2}\cdot 2=\frac{1}{2}\cdot 6+1=4$$

$m_{02}=m_{20}=4$ [analogia jak w przypadku $m_{01}=m_{10}$, dodatkowo sprawdzone w Wolframie bo już nie chciało mi się liczyć]

Zadanie 5.

		X	0	1	2	3
		$f_X(X)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
	$f_Y(Y)$					
0	$\frac{1}{2}$		$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
3	$\frac{1}{6}$		$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{24}$
4	$\frac{1}{6}$		$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{24}$
5	$\frac{1}{6}$		$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{24}$